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A new statistical distribution for characterizing the random strength of brittle materials

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A new three-parameter statistical distribution is offered for the description of random strength of a brittle material. The distribution allows characterization of a wide range of relations regarding the strength–size effect. Thus, in contrast to the Weibull distribution, the non-linear character between the logarithm of average strength and the logarithm of the specimen size may be described, while retaining the Weibull function as a limiting case. Furthermore, the proposed distribution permits simple evaluation of the necessary statistical parameters by considering all experimental points for different specimen sizes together. Experimental confirmation of the strength distribution is illustrated using experimental data on three types of glass fibres at three gauge lengths each.

1. Introduction

The effect of size on the average strength of a brittle material can be explained by the stochastic nature of the material strength [1]. The most widely used distribution function for characterizing the relationships between the variability of strength, the average strength, and the material (specimen) size is the two-parameter Weibull distribution in the following form [1]

$$P(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{A}\right)^{\beta}\right] \quad (1)$$

Where P is the cumulative probability function of strength, σ. The constants A and β are the scale and shape parameters, respectively, V is the specimen volume, and V_0 is the reference volume. The physical interpretation of Equation 1 is that the flaw distribution per unit volume is independent of specimen size. Thus, the volume dependence of the average strength, $\bar{\sigma}$, may be characterized by the shape parameter, β, only

$$\bar{\sigma}/A = \Gamma(1 + 1/\beta)\left(\frac{V}{V_0}\right)^{-1/\beta} \quad (2)$$

While Equation 2 has been shown to be a good approximation of the effect of size on the average strength of many brittle materials, it is not a good representation for all materials over a wide range of sizes. This phenomenon is especially significant, because it may result in a serious strength overestimation of specimens (material elements) of very small size. For example, the strength of very short fibre fragments is of particular interest with regard to the properties of fibre-reinforced composite materials where the critical fibre lengths are in the range 0.1–0.5 mm [2]. Schmitz and Metcalfe [3] showed that significant overestimation of the strength of very short glass fibres occurs when extrapolating data from fibre lengths in the range 2.5–20 cm using a classical Weibull distribution. They suggested that the flaw distribution changed with fibre length and that in order properly to characterize the strength–length relation it was necessary to use different flaw distributions at different lengths. The "weakest link" concept for analysis of the fibre size effect has also been used by Rosen [4]. The strength distribution of an initial element was chosen in the bimodal form of a double rectangular function. A similar idea has been utilized by Fraser *et al.* [5] to estimate the strength of glass fibres at very short lengths.

A modification of Equation 1 is the well-known three-parameter Weibull distribution

$$P(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma - \sigma_0}{A}\right)^{\beta}\right] \quad (3)$$

In this case, the variability of β still determines the effect of size on the "shifted" average strength, $(\sigma - \sigma_0)$.

Other approaches to fitting empirically statistical data on the strength distribution of brittle materials have depended on the use of three- or four-parameter distributions. For example, a factor, α $(0 < \alpha \leqslant 1)$, has been offered for fibres and/or composites in the following form [6, 7]

$$P(\sigma, V) = 1 - \exp\left[-\left(\frac{V}{V_0}\right)^{\alpha}\left(\frac{\sigma}{A}\right)^{\beta}\right] \quad (4)$$

It has been assumed that this factor reflects a difference between fibre strength variability along and across a fibre. The advantages of using Equation 4 have been considered, for example, by Beyerlein and Phoenix [8]. Other modifications of a Weibull distribution have been proposed by Stoner *et al.* [9]

$$P(\sigma, V) = 1 - \exp\left[-\left(\frac{V}{V_0}\right)\left(\frac{\sigma}{A_1}\right)^{\beta_1} - \left(\frac{\sigma}{A_2}\right)^{\beta_2}\right] \quad (5)$$

by Padgett *et al.* [10]

$$P(\sigma, V) = 1 - \exp\left[-\left(\frac{V}{V_0} + \gamma\right)\left(\frac{\sigma}{A}\right)^{\beta}\right] \quad (6)$$

and by Ibnabdeljalil *et al.* [11]

$$P(\sigma, V) = 1 - \exp\left\{-\frac{V}{V_0}\left[\left(\frac{\sigma}{A}\right)^{\beta} + \lambda\right]\right\} \quad (7)$$

Where A_1, A_2, β_1, β_2, γ, λ are some experimentally evaluated parameters. Equation 5 takes into account an end-effect failure mode [9], while Equation 6 assumes a general linear dependence of the characteristic $A^{-\beta}$ on the gauge length, and Equation 7 takes into consideration the possibility of initial discontinuities before loading. Although Equations 4–7 do permit one to take into account observed divergence from the Weibull relation (Equation 2), their application is limited by difficulty in evaluation of their statistical parameters due to the complex form of the exponential terms.

Therefore, the purpose of this work was to offer a new distribution function that could account for a wide range of functional forms of the effect of size on the strength distribution, while retaining a simple method of evaluation of the basic statistical parameters. The utility of the proposed distribution will be illustrated using experimental data on a series of glass fibres with different gauge lengths.

2. Analysis of the distribution

In a more general case, the Weibull distribution may be represented as

$$P(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}F(\sigma)\right] \quad (8)$$

where $F(\sigma)$ is a monotonically increasing function of σ with the only limitation $F(\sigma) \geqslant 0$. Although the physical significance of the expression $F(\sigma)/V_0$ is the number of critical defects per unit volume, the power law approximation of the function $F(\sigma)$ in Equation 1 is justified exclusively by successful representation of experimental data. Therefore, in order to have a more flexible form of $F(\sigma)$, while retaining the utility of the Weibull approach, we propose the following three-parameter approximation

$$F(\sigma) = \left(\frac{\sigma}{A}\right)^{\beta}\exp\left(\frac{\sigma}{B}\right) \quad (9)$$

and the corresponding distribution in a form

$$P(\sigma, V) = 1 - \exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{A}\right)^{\beta}\exp\left(\frac{\sigma}{B}\right)\right] \quad (10)$$

A probability density function, p, may be calculated as ($B \neq 0$)

$$p(\sigma, V) = \frac{\partial P(\sigma, V)}{\partial\sigma} = \frac{V}{V_0}\left(\frac{\sigma}{A}\right)^{\beta}\exp\left(\frac{\sigma}{B}\right)\left[\frac{\beta}{\sigma} + \frac{1}{B}\right] \times\exp\left[-\frac{V}{V_0}\left(\frac{\sigma}{A}\right)^{\beta}\exp\left(\frac{\sigma}{B}\right)\right] \quad (11)$$

As $B \to \infty$, $-\infty$, Equation 10 reduces to the classical two-parameter Weibull distribution (Equation 1). However, at finite magnitudes of parameter B, a non-linear relation between the logarithm of the average strength and the logarithm of specimen size may be taken into account. Let us consider a plot of hypothetical experimental data $\ln(\bar{\sigma}) \rightleftharpoons \ln(V/V_0)$ (Fig. 1a), where $\bar{\sigma}$ is the average strength. A linear dependence is obtained when $B \to \infty$, $-\infty$, while non-linear curves will reflect finite magnitudes of B. The smaller the value of parameter B, the more non-linear is the relationship. The magnitudes of B can be, in general, greater or less than zero: the sign of B indicates the shape (sign of curvature) of the dependence $\ln(\bar{\sigma}) \rightleftharpoons \ln(V/V_0)$ (Fig. 1a). Consequently, experimentally observed non-linearity of the plot $\ln(\bar{\sigma}) \rightleftharpoons \ln(V/V_0)$ may be described by Equation 10. Similarly, for a cumulative probability function for a single size, V, the plot $\ln\{\ln[1/(1-P)]\} \rightleftharpoons \ln(\sigma)$ will be linear when $B \to \infty$, $-\infty$, but non-linear at finite magnitudes of B (Fig. 1b).

From the viewpoint of practical applications, the main statistical characteristics of the distribution, namely the average strength, $\bar{\sigma}$, and standard deviation, σ_{σ}, are of particular interest. These characteristics are calculated as

$$\bar{\sigma} = \int_0^{\infty}\frac{\partial P(\sigma, V)}{\partial\sigma}\sigma\, d\sigma = B\Psi_1(\beta, \varpi) \quad (12)$$

$$\sigma_{\sigma}^2 = \int_0^{\infty}\frac{\partial P(\sigma, V)}{\partial\sigma}(\sigma - \bar{\sigma})^2\, d\sigma = B^2\Psi_2(\beta, \varpi) - \bar{\sigma}^2 \quad (13)$$

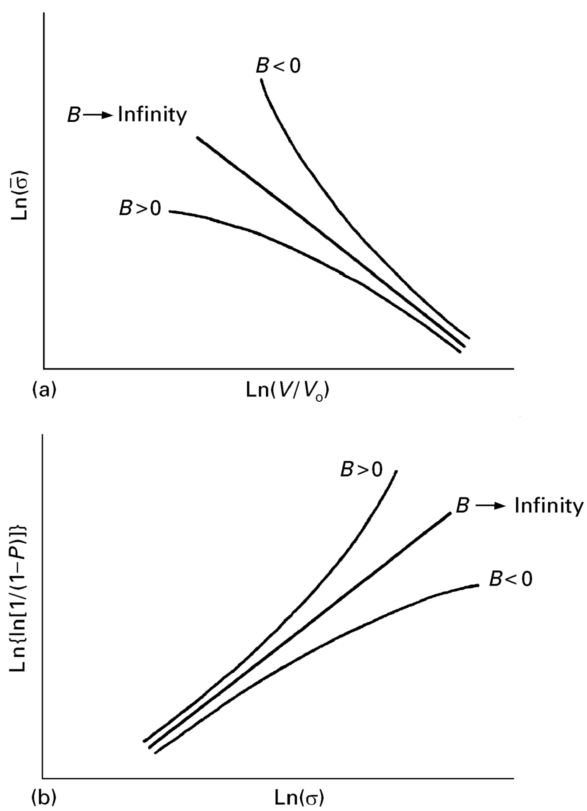

Figure 1 Schematic representation of the distribution.

where $\Psi_1(\beta, \varpi)$, $\Psi_2(\beta, \varpi)$ are dimensionless functions in the following form

$$\Psi_1(\beta, \varpi) = \int_0^\infty p'(\beta, \varpi, \sigma')\,\sigma'\,d\sigma' \qquad (14)$$

$$\Psi_2(\beta, \varpi) = \int_0^\infty p'(\beta, \varpi, \sigma')(\sigma')^2\,d\sigma' \qquad (15)$$

and

$$p'(\beta, \varpi, \sigma') = \varpi(\sigma')^\beta \exp(\sigma')\left(\frac{\beta}{\sigma'} + 1\right) \times \exp[-\varpi(\sigma')^\beta \exp(\sigma')] \qquad (16a)$$

where

$$\varpi = \left(\frac{V}{V_0}\right)\left(\frac{B}{A}\right)^\beta \qquad (16b)$$

and

$$\sigma' = \sigma/B. \qquad (16c)$$

Although functions $\Psi_1(\beta, \varpi)$ and $\Psi_2(\beta, \varpi)$ cannot be represented in an analytical form, their numerical integration may be easily carried out using known magnitudes of β and ϖ.

The proposed distribution is a suitable for numerical analysis at any $B > 0$. However, at $B < 0$, a limitation regarding the monotonic character of $F(\sigma)$ should be noted. Solving for $dF(\sigma)/d\sigma = 0$, one finds that when $B < 0$ the monotonic character of $F(\sigma)$ exists in the range $\sigma < |B|\beta$. It will be shown that in the cases studied where $B < 0$, this condition is fulfilled for the experimentally observed data with a considerable reserve.

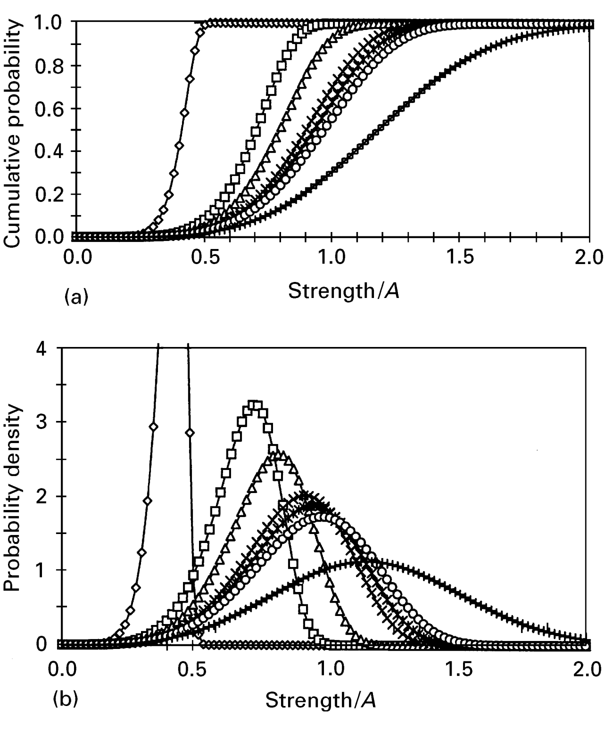

Figure 2 Dependencies of (a) cumulative probability and (b) probability density on ratio B/A at $\beta = 5$ and $k = V/V_0 = 1$. B/A: (◇) 0.1, (□) 0.5, (△) 1, (×) 5, (∗) infinity, (○) −5, (+) −1.

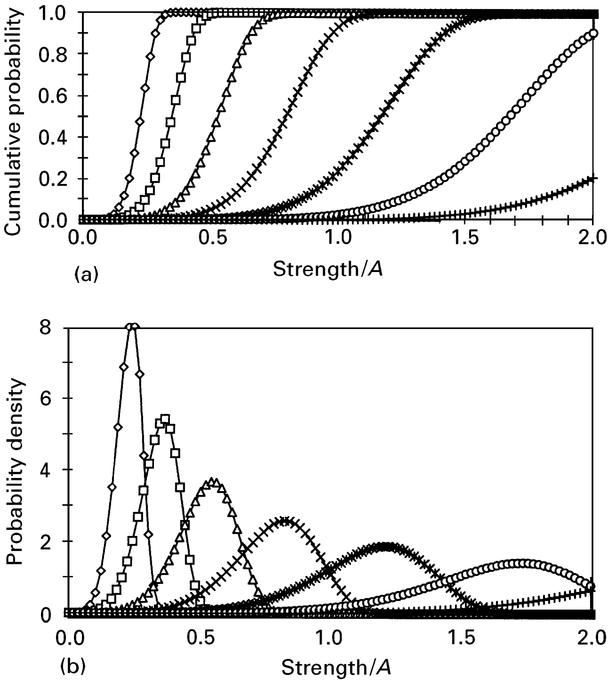

Figure 3 Dependencies of (a) cumulative probability and (b) probability density on $k = V/V_0$ at $\beta = 5$ and $B/A = 1$. k: (◇) 1000, (□) 100, (△) 10, (×) 1, (∗) 0.1, (○) 0.01, (+) 0.001.

Fig. 2 illustrates the effect of the ratio, B/A, on the distribution of relative strength, σ/A. One can note that in contrast with the Weibull distribution ($B \to \infty$, $-\infty$), there is a very convenient opportunity to "manage" the average strength and the strength variability via the ratio B/A. The effect of the reduced size, $k = V/V_0$, on the strength distribution for a fixed value of $B/A = 1$ is illustrated in Fig. 3.

3. Statistical evaluation of the parameters

Values of the parameters β, A and B can be obtained using a statistical treatment of the relevant experimental data. For evaluation of the parameters, we utilize the least square method. This approach allows one to consider all experimental points for specimens with different sizes together. Let us consider results of experimental testing at different specimen volumes V_j, $j = 1, \dots, m$, where m is the number of distinct sizes. The magnitudes of the strength within each jth size, $\sigma_{i,j}$; $i = 1, \dots, n_j$, may be ranked in an increasing order ($\sigma_{i,j} \geqslant \sigma_{i-1,j}$) (where n_j is the number of experimental measurements for jth size). Considering each size separately, one may estimate a cumulative probability $P_{i,j}$ of a failure at $\sigma \leqslant \sigma_{i,j}$ as

$$P_{i,j} = (i - 0.5)/n_j \qquad (17)$$

Consequently, taking twice the logarithm of both sides of Equation 10 and using the above-mentioned notation, one can obtain for each experimental point

$$\ln\{\ln[1/(1 - P_{i,j})]\} = \beta\ln(\sigma_{i,j}) + (1/B)\sigma_{i,j} - \beta\ln(A) + \ln(k_j) \qquad (18)$$

where $k_j = V_j/V_0$. This expression may also be written as

$$y_{i,j} = a_0 + a_1 x1_{i,j} + a_2 x2_{i,j} \tag{19}$$

where $y_{i,j} = \ln\{\ln[1/(1 - P_{i,j})]\} - \ln(k_j)$; $x1_{i,j} = \ln(\sigma_{i,j})$; $x2_{i,j} = \sigma_{i,j}$; $a_0 = -\beta \ln(A)$; $a_1 = \beta$; $a_2 = (1/B)$. For a population containing only one size, Equation 19 is reduced to a simpler form, because at $V = V_0$, $k = 1$, and, therefore, $\ln(k) = 0$. A minimization procedure using the least square method should be used for evaluation of parameters a_0, a_1, a_2 as

$$\Phi = \frac{1}{n} \sum_{j=1}^{m} \sum_{i=1}^{n_j} \{y_{i,j} - a_0 - a_1 x1_{i,j} - a_2 x2_{i,j}\}^2 \to \min \tag{20}$$

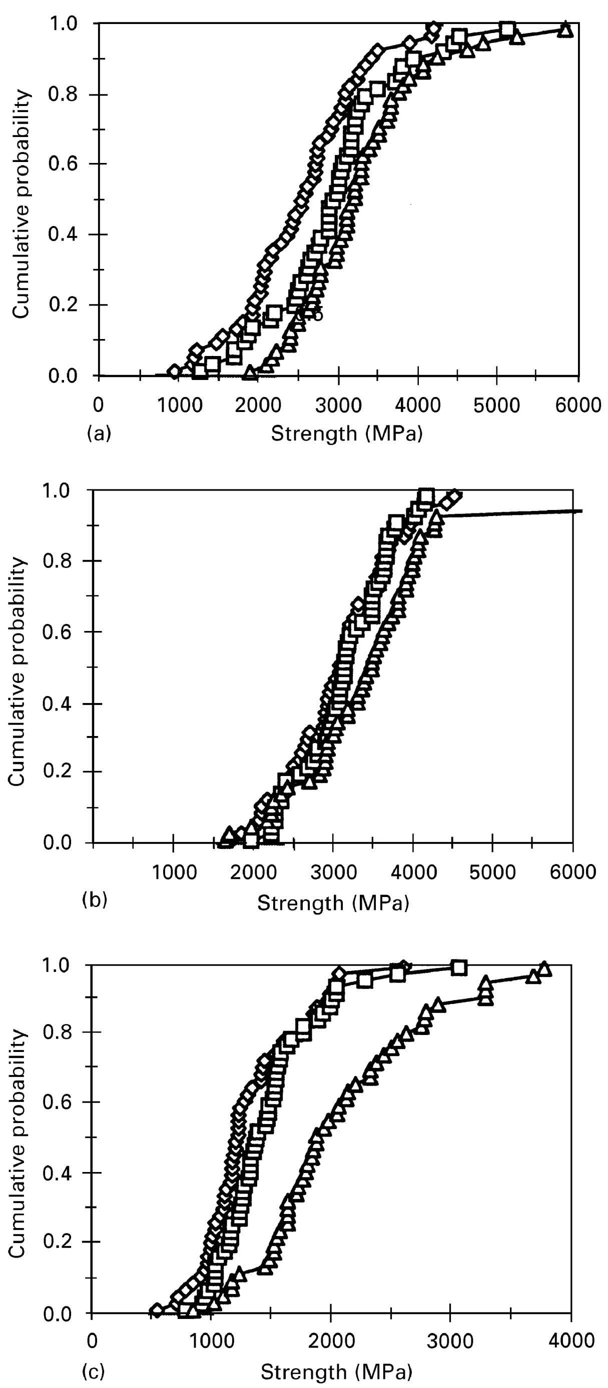

Figure 4 Experimental distribution of (a) "Epoxy", (b) "PEEK", and (c) "Starch" fibres. k: (◇) 1, (□) 0.5, (△) 0.2.

which is reduced to a system of linear equations with a trivial solution

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} n & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x1_{i,j} & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x2_{i,j} \\ \sum_{j=1}^{m}\sum_{i=1}^{n_j} x1_{i,j} & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x1^2_{i,j} & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x1_{i,j}x2_{i,j} \\ \sum_{j=1}^{m}\sum_{i=1}^{n_j} x2_{i,j} & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x1_{i,j}x2_{i,j} & \sum_{j=1}^{m}\sum_{i=1}^{n_j} x2^2_{i,j} \end{bmatrix}^{-1} \begin{bmatrix} \sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{i,j} \\ \sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{i,j}x1_{i,j} \\ \sum_{j=1}^{m}\sum_{i=1}^{n_j} y_{i,j}x2_{i,j} \end{bmatrix} \tag{21}$$

where $n = \sum_{j=1}^{m} n_j$ is the total number of considered experimental points. The desired parameters are evaluated as $\beta = a_1$; $A = \exp(-a_0/\beta)$; $B = 1/a_2$.

4. Experimental analysis

Experimental confirmation of the proposed distribution has been illustrated using experimental strength data on three types of S-2 glass fibres, taken from rovings supplied by Owens Corning: (a) 463 AA epoxy and thermoset compatible sizing, referred to as "Epoxy" coated; (b) 933 AA PEEK and thermoplastic compatible sizing ("PEEK"); and (c) starch-coated fibres ("Starch"). Single strands of fibre were separated from the rovings and mounted on to C-shaped paper templates using scotch tape and fast-curing

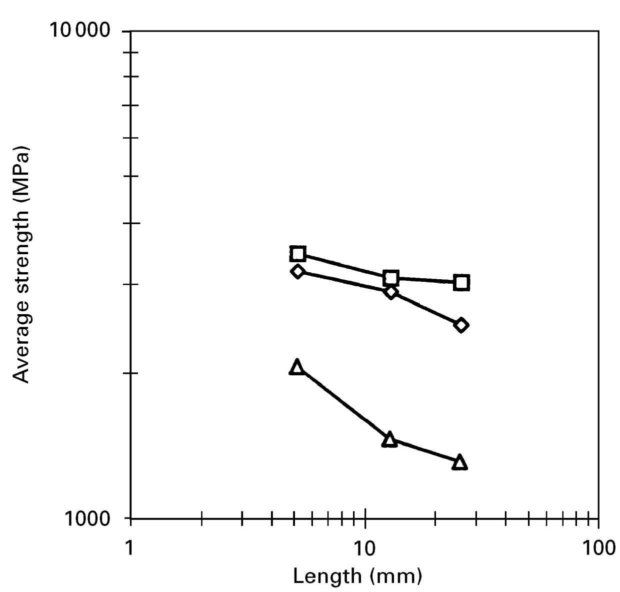

Figure 5 Experimental dependencies of the average strength of fibre length, for (◇) epoxy, (□) PEEK, and (△) starch.

epoxy. These were then pulled in tension in an MTS tensiometer with a 2 1b (~0.9 kg) load cell and the load–extension plot recorded on a chart recorder. Tensile strength was measured at three different gauge lengths, $l = 25.4, 12.7, 5.08$ mm ($k_1 = 1$, $k_2 = 0.5$, $k_3 = 0.2$, respectively) for each type of fibre; approximately 50 tests were performed at each gauge length.

TABLE I Statistical parameters of strength distributions at the reference length 25.4 mm

Fibre	n	[a]	β	A (MPa)	B (MPa)	Φ
Epoxy	146	W	4.52	2644.8	–	0.1216
		P	4.66	2564.0	− 19 148.6	0.1215
PEEK	158	W	5.32	3009.9	–	0.3400
		P	9.13	2045.4	− 877.5	0.2796
Starch	153	W	3.86	1434.5	–	0.1483
		P	7.03	949.0	− 524.6	0.0776

[a] W, Weibull distribution; P, the proposed distribution.

The experimental cumulative distributions of strength are presented in Fig. 4, and the dependence of the average strength on the fibre length is shown in Fig. 5 on logarithmic axes. The non-linearity of the $\ln(\bar{\sigma}) \rightleftharpoons \ln(k)$ plot (i.e. the divergence from the classical Weibull relationship) can be characterized by the parameter B, using the proposed numerical procedure. Calculated values of the parameters β, A, B and the residual Φ are presented in Table I. The residual parameter, Φ, is determined using Equation 20. For calculations using the classical Weibull function, we assume $a_2 = 0$ (because $B \to \infty$). The results for the three fibres reflect different degrees of divergence from a classical Weibull distribution. In the case of epoxy-coated S-glass fibres, the value of B is relatively high

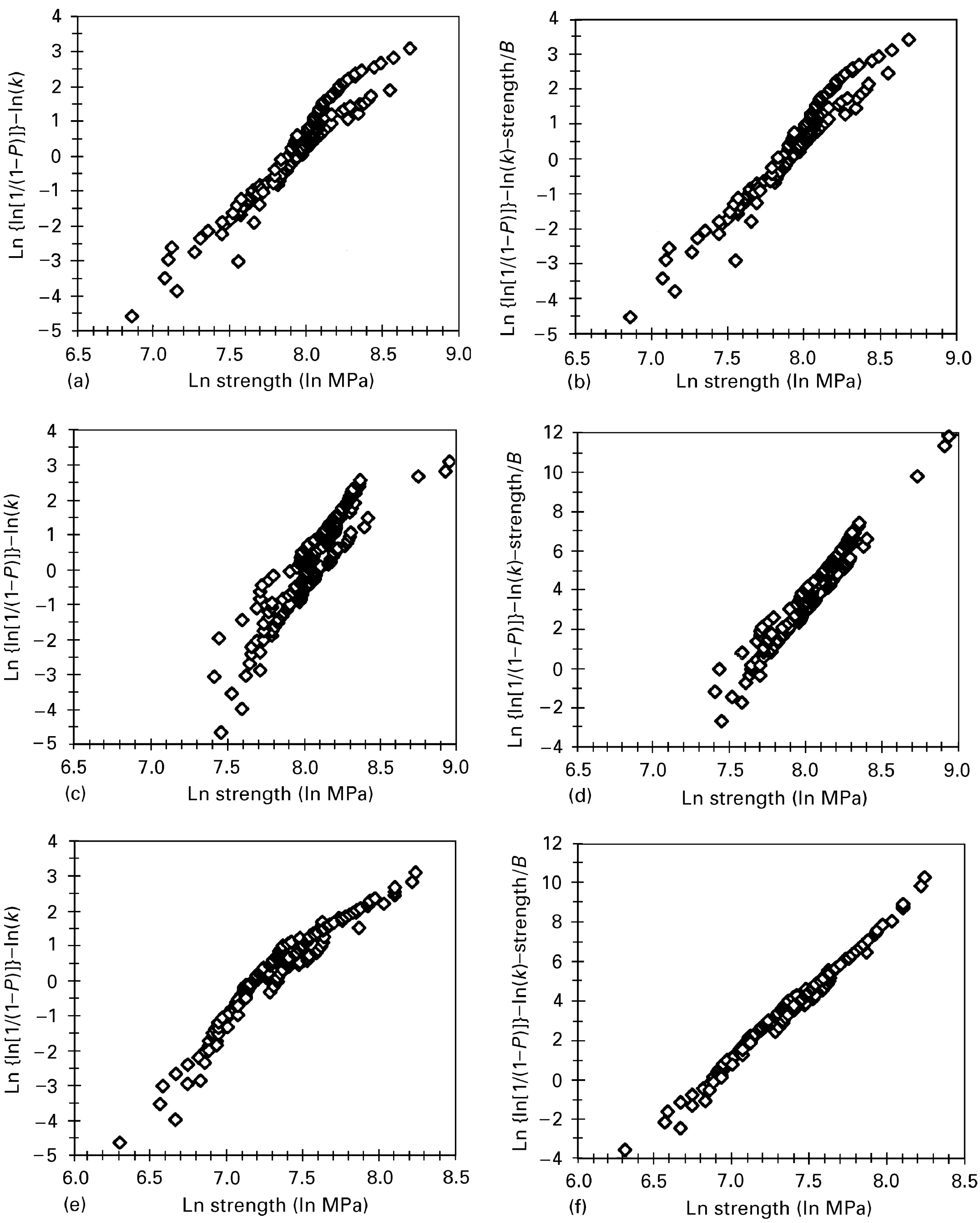

Figure 6 Experimental plot of $y_{i,j} \rightleftharpoons x_{i,j}$ for (a, b) "Epoxy", (c, d) "PEEK", and (e, f) "Starch" fibres, using the (a, c, e) Weibull and (b, d, f) proposed distributions.

($B = -19\,148.6$; $B/A = -7.47$) and, consequently, there is very little difference in A and β in comparison to those for the classical Weibull expression. The difference between respective shape and scale parameters does not exceed 3%. Moreover, the proposed three-parameter distribution provides the same residual ($\Phi = 0.1215$) as that for the two-parameter distribution ($\Phi = 0.1216$). The starch-coated S-glass fibres, however, are better represented by a smaller value of $B = -524.6$ ($B/A = -0.55$). The values of A and β are significantly different for the two models, and application of the proposed distribution results in a residual Φ which is approximately half that for the classical Weibull distribution (0.0776 compared to 0.1438). A similar effect is shown for PEEK-coated S-glass fibres: $B/A = -0.43$ and the residual Φ is reduced from 0.3400 to 0.2796. Graphical confirmations of the above-mentioned results are shown in Fig. 6 as well. Plots of $y_{i,j} \rightleftharpoons x_{i,j}$ for each type of fibre are presented, where $x_{i,j} = \ln(\sigma_{i,j})$ and $y_{i,j} = \ln\{\ln[1/(1 - P_{i,j})]\} - \ln(k_j)$ (plot of the classical of Weibull distribution) or $y_{i,j} = \ln\{\ln[1/(1 - P_{i,j})]\} - \ln(k_j) - \sigma_{i,j}/B$ (plot of the proposed distribution). The difference between the two plots for the epoxy-coated fibres is negligible (Fig. 6a, b), because the effect of parameter B is very small. However, for the PEEK-coated and starch-coated S-glass fibres (Fig. 6c–f), one can see a strong difference between the two distributions. The obvious linear character of Figs 6d and f in contrast with the non-linear dependencies of Figs 6c and e, illustrate the advantage of the proposed distribution.

5. Conclusions

1. The proposed distribution allows characterization of the effect of size on the average strength of brittle materials when there is a non-linear relationship between the logarithm of the strength and the logarithm of the specimen size. The non-linearity of the dependence $\ln(\bar{\sigma}) \rightleftharpoons \ln(V/V_0)$ is taken into account by introducing a third supplementary parameter, $-\infty \leqslant B \leqslant \infty$. In the limits $B \to \infty$, $-\infty$, the distribution reduces to a classical Weibull form.

2. The proposed approach of statistical evaluation of the distribution parameters, β, A, B, permits all experimental points at different specimen sizes to be considered together. Therefore, the calculated parameters possess higher statistical reliability than those obtained using a limited number of average values only.

3. It is shown that in contrast with a classical Weibull distribution, utilization of the proposed distribution allows the residual parameter, Φ, to be reduced significantly for experimental data regarding strength distribution at different specimen sizes (gauge lengths).

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